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Note

A class of nonlinear equations in Hilbert space and its application to completeness problems

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Abstract

Nonlinear equations arising in the spectral theory of self-adjoint operator functions and related completeness problems for eigenvectors are studied. A separation theorem about the values of the Rayleigh functional on solutions of a nonlinear equation is proved. This theorem is used, as a new approach to establish completeness of eigenvectors for some classes of self-adjoint operator functions. Examples from matrix pencils are given.

Keywords: Resolvent; Operator functions; Eigenvectors; Eigenvalues

1. Introduction

Denote by $C^1([a,b], S(H))$ the class of self-adjoint and continuously differentiable operator functions defined on an interval $[a,b] \subset R^1$ whose values are in the space of self-adjoint bounded operators, denoted here by $S(H)$. We suppose:

(I) The Hilbert space H has a decomposition into disjoint cones of the form $H = H_0 \sqcup H_\emptyset$ (we suppose $0 \in H_0$) such that, for all $0 \neq x \in H_0$ the function $\varphi_x(\lambda) := (L(\lambda)x, x)$ has a simple zero $p(x)$ in $[a,b]$ and $(L'(p(x))x, x) > 0$.
(II) $\varphi_x(\lambda) > 0$ for all $x \in H_\emptyset$ and $\lambda \in [a,b]$.

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In the sequel we assume that the conditions (I) and (II) are satisfied. Thus we have a Rayleigh functional $p(x)$ defined only on the conic set H_0. The nonlinear operator defined as

$$Tx = \begin{cases} L(p(x))x, & 0 \neq x \in H_0, \\ 0, & x = 0, \end{cases}$$

solvability problems for the nonlinear equations $Tx = y$ and establishing a connection between these problems and completeness problems are the main subject of this article.

Note that the pair $\{L, p\}$ is often called a Rayleigh system and for strong Rayleigh systems, in addition it is supposed that $H = H_0$, i.e., $H_\emptyset = \emptyset$ (see [1,5,7]). An immediate corollary of the condition (I) is

Corollary 1.1. *If $x \in H_0$ then $\varphi_x(\lambda) < 0$ $(> 0, = 0)$ if and only if $p(x) > \lambda$ $(< \lambda, = \lambda)$.*

It is the aim of this article, as mentioned above to study questions about completeness of eigenvectors of self-adjoint operator functions, corresponding to eigenvalues from $[a, b]$. Similar problems were studied in [2,4,7–10]. In these papers particularly, the following class of operator functions was studied:

(i) $L(\lambda)$ is an analytic or is a function from $C^k([a, b], S(H))$, such that $L(a) \ll 0$, $L(b) \gg 0$, for all $x \in H \setminus \{0\}$,
(ii) the function $(L(\lambda)x, x)$ has exactly one zero in (a, b) and $\sigma_{\mathrm{ess}}(L) = \{\gamma\} \in (a, b)$, where $\sigma_{\mathrm{ess}}(L)$ denotes the essential spectrum.

Under these conditions theorems, establishing the fact that eigenvectors of $L(\lambda)$ corresponding to eigenvalues in (a, b) form a Riesz basis for the Hilbert space, were proved in the above mentioned papers. Namely, such a theorem in the case $k = 1$ in the finite dimensional space was proved in [1]. For analytic operator functions a result was given in [7] (see Theorem 30.12) and the proof is based on a representation of the form

$$L(\lambda) = M(\lambda)(\lambda I - Z), \tag{1.1}$$

where $M(\lambda)$ is invertible on $[a, b]$, $\sigma(Z) \subset (a, b)$ and the operator Z is similar to a self-adjoint operator.

In [8,9] (see also [10]) the formula (1.1) was established in the case $k = 2$ but under the following additional condition: $\int_0^{t_0} \frac{w(t, L'')}{t}\, dt < +\infty$ for sufficiently small t_0, where $w(t, L'')$ is the modulus of continuity for L''. In [4] we recently studied such kind of problems in a dense subset of $C([a, b], S(H))$.

Now we give some needed definitions. The spectrum $\sigma(L)$, the point spectrum or the set of eigenvalues $\sigma_e(L)$ of L are subsets of $[a, b]$ defined as follows: $\lambda \in \sigma(L)$ if $0 \in \sigma(L(\lambda))$ and $\lambda \in \sigma_e(L)$ if $0 \in \sigma_e(L(\lambda))$. A nonzero vector x from the kernel $\ker L(\lambda)$ for $\lambda \in \sigma_e(L)$ is called an eigenvector of L corresponding to λ. The limit spectrum $\pi(L)$ is defined as $\pi(L) = \{\lambda \in (a, b) \mid \exists x_n,\ \|x_n\| = 1,\ x_n \to 0 \text{ (weakly)},\ L(\lambda)x_n \to 0\}$. Denote by $\rho(L) := \{\lambda \in [a, b] \mid L^{-1}(\lambda)$ is boundedly invertible$\}$ the resolvent set of the operator function $L(\lambda)$ and set $R(\lambda) := L^{-1}(\lambda)$, $\lambda \in \rho(\lambda)$, which is called the resolvent of $L(\lambda)$.

In Section 2 we extend the resolvent to isolated eigenvalues and use the extended resolvent to construct some operator and numerical functions. Our main idea is based on properties of these functions. For this purpose we first prove a theorem (Theorem 2.1) about the expansion of the resolvent in a neighborhood of an isolated eigenvalue. In Section 3 we define a nonlinear operator T and study some solvability problems for the equation $Tx = y$. We make in this section some conclusions about completeness of eigenvectors. Concrete examples are given at the end of the paper.

2. An extension of the resolvent to isolated eigenvalues and related special functions

In this section we extend the resolvent to isolated eigenvalues. For this we first prove a theorem on the expansion of the resolvent $R(\lambda)$ around an isolated eigenvalue in $[a,b]$. Then we use this expansion to construct the needed extension of the resolvent and some special functions which play important roles in the sequel.

Theorem 2.1. *Let $L(\lambda)$ be an operator function from $C^1([a,b], S(H))$ satisfying the conditions* (I) *and* (II). *If $\lambda_0 \in \sigma(L) \setminus \pi(L)$ then λ_0 is an isolated eigenvalue of finite multiplicity. Moreover, λ_0 is a simple pole of $R(\lambda)$, that is,*

$$R(\lambda) = \frac{P(\lambda_0)}{\lambda - \lambda_0} A(\lambda) + B(\lambda), \tag{2.1}$$

where $P(\lambda_0)$ is the projection on $\operatorname{Ker}(L(\lambda_0))$, *$A(\lambda)$ and $B(\lambda)$ are continuous in a neighborhood of λ_0.*

Proof. It is known (see [1, Proposition 3.7] and [5]) that a number from $\sigma(L) \setminus \pi(L)$ is an isolated eigenvalue of finite multiplicity. Set $L_1(\lambda) := L(\lambda) + \lambda I$. Evidently, $\lambda_0 \in \sigma_e(L) \Leftrightarrow \lambda_0 \in \sigma_e(L_1(\lambda_0))$, where by $\sigma_e(L_1(\lambda_0))$ we denote the set of eigenvalues of the operator $L_1(\lambda_0)$. Consequently, λ_0 should be an isolated eigenvalue of the self-adjoint operator $L_1(\lambda_0)$. Now define the forms

$$\Delta[\alpha,\beta] = \begin{cases} \frac{L(\alpha)-L(\beta)}{\alpha-\beta}, & \alpha \neq \beta, \\ L'(\alpha), & \alpha = \beta, \end{cases} \quad \text{and} \quad \Delta[\alpha] := \Delta[\alpha,\alpha] = L'(\alpha).$$

Then we have $\Delta_1[\alpha,\beta] = \Delta[\alpha,\beta] + I$ and $\Delta_1[\alpha] = \Delta[\alpha] + I$, where $\Delta_1[\alpha,\beta]$ is the form for the operator function $L_1(\lambda)$. By the condition $L_1(\lambda_0)$ is a self-adjoint operator and for the resolvent R_λ^0 of this operator the following formula

$$R_\lambda^0 = \frac{P(\lambda_0)}{\lambda - \lambda_0} + Q(\lambda) \tag{2.2}$$

holds. Recall that $R_\lambda^0 = (L_1(\lambda_0) - \lambda I)^{-1}$, and $Q(\lambda)$ is analytic in a neighborhood of λ_0 (see [3]).

$$\begin{aligned} L^{-1}(\lambda) &= \bigl(L_1(\lambda) - \lambda I\bigr)^{-1} = \bigl(L_1(\lambda_0) - \lambda I + L_1(\lambda) - L_1(\lambda_0)\bigr)^{-1} \\ &= \bigl[\bigl(I + \bigl(L_1(\lambda) - L_1(\lambda_0)\bigr)R_\lambda^0\bigr)\bigl(L_1(\lambda_0) - \lambda I\bigr)\bigr]^{-1} \\ &= R_\lambda^0\bigl[I + \bigl(L_1(\lambda) - L(\lambda_0)\bigr)R_\lambda^0\bigr]^{-1}. \end{aligned} \tag{2.3}$$

Let

$$H(\lambda) = I + \bigl(L_1(\lambda) - L_1(\lambda_0)\bigr)R_\lambda^0.$$

Now, we are going to prove that $H(\lambda)$ is invertible in a full neighborhood of λ_0. By (2.2),

$$\begin{aligned} H(\lambda) &= I + \bigl(L_1(\lambda) - L_1(\lambda_0)\bigr)R_\lambda^0 = I + \bigl(L_1(\lambda) - L_1(\lambda_0)\bigr)\left(\frac{P(\lambda_0)}{\lambda - \lambda_0} + Q(\lambda)\right) \\ &= I + \Delta_1(\lambda,\lambda_0)P(\lambda_0) + \bigl(L_1(\lambda) - L_1(\lambda_0)\bigr)Q(\lambda). \end{aligned}$$

Hence

$$H(\lambda_0) = I + \Delta_1(\lambda_0)P(\lambda_0)$$

and $P(\lambda_0)$ is compact as an operator of finite rank. Thus $\Delta_1(\lambda_0)P(\lambda_0)$ is compact. If $H(\lambda_0)$ is not invertible then λ_0 is an eigenvalue of $H(\lambda_0)$. It means that $H(\lambda_0)y = 0$ for some $y \neq 0$,

$$H(\lambda_0)y = y + \Delta_1(\lambda_0)P(\lambda_0)y = y + \big(L'(\lambda_0) + I\big)P(\lambda_0)y = 0.$$

But then

$$\big(P(\lambda_0)y, H(\lambda_0)y\big) = 0$$

and

$$0 = \big(P(\lambda_0)y, y\big) + \big(L'(\lambda_0)P(\lambda_0)y, P(\lambda_0)y\big) + \big(P(\lambda_0)y, P(\lambda_0)y\big).$$

Since $P(\lambda_0)y \in \operatorname{Ker}(L(\lambda_0))$ then $(L'(\lambda_0)P(\lambda_0)y, P(\lambda_0)y) > 0$. Thus $P(\lambda_0)y = 0$.

Finally, using the equation $H(\lambda_0) = I + \Delta_1(\lambda_0)P(\lambda_0)$ we obtain $y = 0$. The contradiction shows that $H(\lambda)$ is invertible in a full neighborhood of λ_0. On the other hand, it follows from (2.3) and (2.2) that

$$R(\lambda) := L^{-1}(\lambda) = \left(\frac{P(\lambda_0)}{\lambda - \lambda_0} + Q(\lambda)\right)H^{-1}(\lambda).$$

Consequently, the assertion of the theorem holds with $A(\lambda) = H^{-1}(\lambda)$ and $B(\lambda) = Q(\lambda)H^{-1}(\lambda)$. □

Now we construct the extension of the resolvent $R(\lambda)$ on $[a,b] \setminus \pi(L)$ as

$$\tilde{R}(\lambda) = \begin{cases} B(\lambda_0), & \lambda = \lambda_0 \in \sigma_e(L), \\ R(\lambda), & \lambda \in \rho(L). \end{cases}$$

Define also the following operator functions:

$$\tilde{A}(\lambda) = \begin{cases} A(\lambda_0), & \lambda = \lambda_0 \in \sigma_e(L), \\ I, & \lambda \in \rho(L), \end{cases}$$

and $F(\lambda) = (I - P(\lambda)\tilde{A}(\lambda)L'(\lambda))\tilde{R}(\lambda)$. The main numerical function is defined by $R(\lambda, x) := (\tilde{R}(\lambda)x, x)$. Note that similar functions were defined in [1, p. 129] for strong Rayleigh systems, which means $H = H_0$ in our case. They were also used there to solve problems different from those considered in our paper. Now we prove a theorem about properties of these functions, which will be used in the next chapter. For this we set $P(\lambda) = 0$ for $\lambda \in \rho(\lambda)$.

Theorem 2.2. *Let $\lambda \in [a,b] \setminus \pi(L)$ and $x \in P^{\perp}(\lambda)$. Then the following relations*

(a) $\tilde{A}(\lambda)x = L(\lambda)\tilde{R}(\lambda)x = L(\lambda)F(\lambda)x = x$,

(b) $$R(\lambda, x) = \big(L(\lambda)F(\lambda)x, F(\lambda)x\big), \tag{2.4}$$

$$R'(\lambda, x) = -\big(L'(\lambda)F(\lambda)x, F(\lambda)x\big) \tag{2.5}$$

hold.

Proof. (a) Let us consider two different cases: $\lambda \in \sigma_e(L)$ and $\lambda \in \rho(L)$. Take first $\lambda \in \rho(L)$. In this case since $\tilde{A}(\lambda) = I$, $\tilde{R}(\lambda) = R(\lambda)$ and $F(\lambda) = R(\lambda)$ we have (a). Now suppose that

$\lambda = \lambda_0 \in \sigma_e(L)$. Then $\tilde{A}(\lambda_0) = A(\lambda_0) = H^{-1}(\lambda_0)$. It follows from $H(\lambda_0) = I + \Delta_1(\lambda_0)P(\lambda_0)$ that $H(\lambda_0)x = x$, $x \in P^{\perp}(\lambda_0)$ and since $H(\lambda_0)$ is invertible then $H^{-1}(\lambda_0)x = x$, $x \in P^{\perp}(\lambda_0)$. Thus $\tilde{A}(\lambda)x = x$. For $\lambda_0 \in \sigma_e(L)$ we have $\tilde{R}(\lambda_0) = B(\lambda_0) = Q(\lambda_0)H^{-1}(\lambda_0)$. By (2.1),

$$R(\lambda) = \frac{P(\lambda_0)}{\lambda - \lambda_0} A(\lambda) + B(\lambda).$$

Applying $L(\lambda)$ to the both sides of this inequality we obtain

$$I = \frac{L(\lambda) - L(\lambda_0)}{\lambda - \lambda_0} P(\lambda_0)A(\lambda) + L(\lambda)B(\lambda).$$

Taking the limit as $\lambda \to \lambda_0$ we can write

$$I = L'(\lambda_0)P(\lambda_0)A(\lambda_0) + L(\lambda_0)B(\lambda_0) = L'(\lambda_0)P(\lambda_0)A(\lambda_0) + L(\lambda_0)\tilde{R}(\lambda_0).$$

Using $A(\lambda_0)x = H^{-1}(\lambda_0)x = x$, $x \in P^{\perp}(\lambda_0)$ we obtain $L'(\lambda_0)P(\lambda_0)A(\lambda_0)x = 0$ for $x \in P^{\perp}(\lambda_0)$. Consequently, $L(\lambda_0)\tilde{R}(\lambda_0)x = x$. Let us prove the last relation in (a): $L(\lambda_0)F(\lambda_0)x = x$. For $\lambda_0 \in \sigma_e(L)$ we have

$$F(\lambda_0) = \bigl(I - P(\lambda_0)A(\lambda_0)L'(\lambda_0)\bigr)B(\lambda_0) = B(\lambda_0) - P(\lambda_0)A(\lambda_0)L'(\lambda_0)B(\lambda_0)$$

and

$$L(\lambda_0)F(\lambda_0) = L(\lambda_0)B(\lambda_0) - L(\lambda_0)P(\lambda_0)A(\lambda_0)L'(\lambda_0)B(\lambda_0).$$

Using

$$L(\lambda_0)P(\lambda_0)A(\lambda_0)L'(\lambda_0)B(\lambda_0)x = 0$$

for $x \in P^{\perp}(\lambda_0)$ we obtain

$$L(\lambda_0)F(\lambda_0)x = L(\lambda_0)B(\lambda_0)x = x,$$

for $x \in P^{\perp}(\lambda_0)$.

(b) We will check only the case $\lambda \in \rho(L)$. By using the definitions of the functions defined above one can check the other case when $\lambda = \lambda_0 \in \sigma_e(L)$. Now for $\lambda \in \rho(L)$ we have $F(\lambda) = R(\lambda)$ and consequently $R(\lambda, x) = (R(\lambda)x, x) = (L(\lambda)F(\lambda)x, F(\lambda)x)$. To prove the second relation we use $R'(\lambda) = -R(\lambda)L'(\lambda)R(\lambda)$ which immediately follows from the following identity for operator functions:

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)\Delta[\lambda, \mu]R(\mu).$$

Thus

$$\bigl(R'(\lambda)x, x\bigr) = \bigl(\tilde{R}(\lambda)x, x\bigr)' = -\bigl(R(\lambda)L'(\lambda)R(\lambda)x, x\bigr) = -\bigl(L'(\lambda)F(\lambda)x, F(\lambda)x\bigr). \qquad \square$$

3. A class of nonlinear equations

Define the following nonlinear operator:

$$Tx = \begin{cases} L(p(x))x, & 0 \neq x \in H_0, \\ 0, & x = 0. \end{cases}$$

Evidently, T is a continuous nonlinear operator defined on H_0. Continuity at zero follows from $\lim_{x \to 0} \|L(p(x))x\| = 0$. Note that we can define the operator T on $H \setminus H_0$ as $Tx = 0$ but for problems we study in this paper it is enough to define T only on H_0.

Set

$$\gamma_- = \inf\{p(x) \mid x \in H_0 \setminus \{0\}\} \quad \text{and} \quad \gamma_+ = \sup\{p(x) \mid x \in H_0 \setminus \{0\}\}.$$

By the condition (I) we have $[\gamma_-, \gamma_+] \subset [a, b]$. Let $\theta = \sup \pi(L)$ and suppose $\theta \in (\gamma_-, \gamma_+)$. Then $\sigma(L) \cap (\theta, \gamma_+]$ consists of eigenvalues of finite multiplicities. Denote by M the span of the eigenvectors, corresponding to the eigenvalues from $(\theta, \gamma_+]$. The main question is whether $\overline{M} = H$? First, we prove a separation theorem.

Theorem 3.1. *Let $0 \neq y \in M^\perp$ and the equation $Tx = y$ has a solution x. Then $p(x) \leqslant \theta$.*

Proof. First we show that for $0 \neq y \in M^\perp$ and $\alpha \in (\theta, \gamma_+]$ the inequality $R(\alpha, y) > 0$ holds. By (2.4), $R(\lambda, y) = (L(\lambda)F(\lambda)y, F(\lambda)y)$ and by the conditions (I) and (II), $L(\gamma_+) \geqslant 0$. Consequently, $R(\gamma_+, y) \geqslant 0$. Let us show that $R(\gamma_+, y) > 0$. Otherwise we obtain $F(\gamma_+)y = 0$ and then it follows from (a) of Theorem 2.2 that $L(\gamma_+)F(\gamma_+)y = y = 0$. But it contradicts to $y \neq 0$. Now suppose that there exists a number $\alpha_0 \in (\theta, \gamma_+)$ such that $R(\alpha_0, y) \leqslant 0$. Then using the continuity of the function $R(\alpha, y)$ we conclude that there are zeros of this function on the interval $[\alpha_0, \gamma_+)$. Denote by t the maximum of these zeros. Then $R(t, y) = 0$ and whence by (2.4), $(L(t)F(t)y, F(t)y) = 0$. Now $F(t)y \in H_0$ and $p(F(t)y) = t$. Hence, by the condition (I), $(L'(t)F(t)y, F(t)y) > 0$ and it follows from this and (2.5) that $R'(t, y) < 0$. Now both $R(t, y) = 0$ and $R'(t, y) < 0$ together contradict to maximality of t. It means that the function $R(\alpha, y)$ should be positive on $(\theta, \gamma_+]$, i.e. $R(\alpha, y) > 0$ for all $\alpha \in (\theta, \gamma_+]$ and $0 \neq y \in M^\perp$. It immediately follows from (2.4) that $R(p(x), L(p(x))x) = 0$ for all $0 \neq x \in H_0$. On the other hand, if $p(x) > \theta$ then by the above proved inequality $R(\alpha, y) > 0$ we can write $R(p(x), L(p(x))x) > 0$. This contradiction means $p(x) \leqslant \theta$. □

Finally, we can make from this separation property the following conclusions. Note that the general case $\pi(L) = \{\gamma\} \in [\gamma_-, \gamma_+]$ is reduced to the case $\pi(L) = \{\gamma_-\}$ by shifting the argument. Namely, we can study the operator function $S(\lambda) := L(\lambda + \gamma - \gamma_-)$ instead of $L(\lambda)$. For this reason in what follows we assume that $\pi(L) = \{\gamma_-\}$ and $H_0 = H$.

Corollary 3.1. *Let $\pi(L) = \{\gamma_-\}$. Then the equation $Tx = y$, $0 \neq y \in M^\perp$, is not solvable.*

Proof. Indeed if $Tx = y$ for some $x \in H$ then by Theorem 3.1 we have $p(x) = \gamma_-$. Clearly, a nonzero vector x satisfying $p(x) = \gamma_-$ or $p(x) = \gamma_+$ is an eigenvector corresponding to the eigenvalue γ_+ or γ_-, respectively. This fact follows from the inequality

$$\|L(\gamma_\pm x)\|^2 \leqslant |L(\gamma_\pm x, x)| \|L(\gamma_\pm)\|.$$

Now we have $Tx = L(p(x))x = 0$ which contradicts to solvability of the equation $Tx = y$, for $0 \neq y \in M^\perp$. □

Corollary 3.2. *If $R(T) = H$ then the eigenvectors corresponding to eigenvalues from $(\gamma_-, \gamma_+]$ are complete in H.*

Proof. If $\overline{M} \neq H$ then $M^\perp \neq \{0\}$. Take $0 \neq y \in M^\perp$. Since $R(T) = H$ then the equation $Tx = y$ is solvable. Thus we have a contradiction with Corollary 3.1. □

Corollary 3.3. *Let $\pi(L) = \gamma_-$ and y be a nonzero vector such that the equation $Tx = y$ is not solvable. If $y \notin M^{\perp}$ then $\overline{M} = H$.*

Finally, notice that there are many operators satisfying $R(T) = H$. In this case we apply Corollary 3.2. To prove $R(T) = H$ we can use a theorem from Nonlinear Functional Analysis [6]. In some cases the equation $Tx = y$ is not solvable if and only if y is an eigenvector or $y \notin M^{\perp}$.

Here we give some finite dimensional examples.

4. Examples

Example 1. Let $L(\lambda) = A - \lambda B$, where $A = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} b & 0 \\ 0 & c \end{pmatrix}$.

Setting $x = (x_1, x_2)$ we have $\langle Ax, x\rangle = ax_1^2 + 2x_1x_2$ and $\langle Bx, x\rangle = bx_1^2 + cx_2^2$. Thus $p(x) = \frac{ax_1^2+2x_1x_2}{bx_1^2+cx_2^2}$. Then the equation $L(p(x))x = Ax - p(x)Bx = y$ can be written in the form

$$\begin{pmatrix} ax_1 + x_2 \\ x_1 \end{pmatrix} - \frac{ax_1^2 + 2x_1x_2}{bx_1^2 + cx_2^2}\begin{pmatrix} bx_1 \\ cx_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

A simple computation gives

$$x_1 = \frac{y_2(by_2^2 + cy_1^2)}{by_2^2 + acy_1y_2 - cy_1^2} \quad \text{and} \quad x_2 = -\frac{y_1(by_2^2 + cy_1^2)}{by_2^2 + acy_1y_2 - cy_1^2}.$$

Now if we choose a, b and c such that $by_2^2 + acy_1y_2 - cy_1^2 > 0$ or $by_2^2 + acy_1y_2 - cy_1^2 < 0$ is satisfied then we have $R(T) = R^2$. For example choosing $c = -1$, $a = 2$ and $b = 2$ we obtain $2y_2^2 - 2y_1y_2 + y_1^2 = y_2^2 + (y_2 - y_1)^2 > 0$.

Note. The operator T is homogeneous. Therefore to investigate the solvability of the nonlinear equation $Tx = y$ it is sufficient to choose $\|y\| = 1$ and use the condition $(x, y) = 0$ which follows from $(L(p(x))x, x) = 0$.

Example 2. Let $L(\lambda) = A - \lambda B$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$.

This is a particular case of Example 1 when $a = 0, b = 2$ and $c = 1$. Now it follows from the expressions of x_1 and x_2, given in Example 1 by using $y_1^2 + y_2^2 = 1$ that

$$x_1 = \frac{y_2(1 + y_2^2)}{2 - 3y_1^2} \quad \text{and} \quad x_2 = -\frac{y_1(1 + y_2^2)}{2 - 3y_1^2}.$$

Now only for vectors y from the subspace spanned by $\begin{pmatrix} \pm\sqrt{\frac{2}{3}} \\ \pm\sqrt{\frac{1}{3}} \end{pmatrix}$ the equation $Tx = y$ does not have a solution. But these are eigenvectors.

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